

# *R*-mode Instability of Slowly Rotating Non-isentropic Relativistic Stars

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We investigate properties of *r*-mode instability in slowly rotating relativistic polytropes. Inside the star slow rotation and low frequency formalism that was mainly developed by Kojima is employed to study axial oscillations restored by Coriolis force. At the stellar surface, in order to take account of gravitational radiation reaction effect, we use a near-zone boundary condition instead of the usually imposed boundary condition for asymptotically flat spacetime. Due to the boundary condition, complex frequencies whose imaginary part represents secular instability are obtained for discrete *r*-mode oscillations in some polytropic models. It is found that such discrete *r*-mode solutions can be obtained only for some restricted polytropic models. Basic properties of the solutions are similar to those obtained by imposing the boundary condition for asymptotically flat spacetime. Our results suggest that existence of a continuous part of spectrum cannot be avoided even when its frequency becomes complex due to the emission of gravitational radiation.

## I. INTRODUCTION

Andersson [1] and Friedman and Morsink [2] discovered that all *r*-modes, which are quasi-toroidal modes mainly restored by Coriolis force, in all rotating stars become unstable due to the gravitational radiation reaction if other dissipative processes are not considered. This instability is clearly understood by the so called CFS mechanism [3–5]. As shown by Lindblom, Owen, and Morsink [6] first, this instability still strongly affects on stability of typical neutron star models even if viscous dissipation of neutron star matter, which tends to stabilize the CFS instability, is taken into account. Since then a lot of studies on oscillation modes restored by Coriolis force in rotating stars have been done to prove their possible importance in astrophysics (for recent review, see, e.g., Refs [7–10]).

Influence of the *r*-mode on stability of rotating neutron stars is one of the most important and interesting phenomenon in astrophysics. In oscillations of neutron stars, relativistic effect must be important because such stars are sufficiently compact. But most studies have been done within the framework of Newtonian gravity so far, although our understandings of *r*-modes have been improved by those investigations. As for *r*-modes studied within the framework of general relativity, Kojima [11] derived master equations for *r*-mode oscillation in the lowest order slow rotation approximation, and then he found possible existence of a continuous part of spectrum in his equations. Beyer and Kokkotas [12] generally verified the existence of a continuous part of spectrum in Kojima's equation. Kojima's formalism was developed to include higher order rotational effects by Kojima and Hosonuma [13,14]. Lockitch, Andersson, and Friedman [15] obtained the discrete *r*-mode solutions in uniform density stars as well as a continuous part of spectrum by solving Kojima's equation. Recently Yoshida [16] and Ruoff and Kokkotas [17] discussed that such discrete *r*-mode solutions are not simply allowed to appear in compressible stellar models. Their results showed that for typical neutron star models, Kojima's equations do not have such a discrete *r*-mode solution.

These recent developments of understanding of relativistic *r*-modes have shown that basic properties of *r*-mode oscillations in relativistic stars are significantly different from those in Newtonian stars. As for non-isentropic stars, most previous studies have shown the existence of a continuous part of spectrum. This is a great contrast with Newtonian case. For Newtonian cases, there are discrete mode solutions and no continuous parts of spectrum for *r*-modes in all uniformly rotating stars as long as their rotation velocity is small enough. It is not however likely to occur such drastic change in the behavior of the solutions due to the inclusion of even a little relativistic effect. Therefore, most authors have considered that a continuous part of spectrum does not appear if some effects that were omitted in previous studies are taken into account. One of such effects is a dissipation effect due to gravitational radiation reaction. In most studies on *r*-modes, a slow motion approximation has been employed in their analysis because of slow rotation approximation. A slow motion approximation changes a wave type equation into a Laplace type equation. Thus, Kojima's equations do not have solutions with wave character, and then the frequency is real number if asymptotically flat spacetime is assumed. In other words, influence of gravitational radiation on relativistic *r*-modes has not been taken into account so far. In this case, Kojima's equation becomes that of singular eigenvalue problem for some frequency range, and hence has a continuous part of spectrum. As suggested by Lockitch et al.

[15] (see also, Ref. [12]), however, Kojima's equation may become that of regular eigenvalue problem if frequency has non-zero imaginary part. Therefore, It is hoped that the existence of a continuous part of spectrum might be avoided if the effect of the emission of the gravitational radiation is considered.

In this paper, accordingly, we will attempt to include the lowest order effect of gravitational radiation reaction into Kojima's formalism. Because Kojima's equation is not wave type equation, as mentioned before, we can not obtain information about gravitational wave from the equation at all. In order to include the gravitational radiation reaction effect into Kojima's equation, we will employ a near-zone boundary condition instead of usually used boundary condition for asymptotically flat spacetime. This boundary condition was introduced by Thorne [18] to include the effect of gravitational radiation reaction on polar pulsations in a Newtonian star. We start, in §2, with the description of our method of solution. A near-zone boundary condition is introduced there in order to take account of the gravitational radiation reaction. In §3, we show the properties of the  $r$ -mode solutions obtained by imposing the boundary condition. §4 is devoted for discussions and conclusions. Throughout this paper we will use units in which  $c = G = 1$ , where  $c$  and  $G$  denote the velocity of light and the gravitational constant, respectively.

## II. METHOD OF SOLUTIONS

We consider slowly rotating relativistic stars with a uniform angular velocity  $\Omega$ , where we take account of the first order rotational effect in  $\Omega$ . The geometry around the equilibrium stars can be described by the following line element (see, e.g. Ref. [19]):

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 - 2\omega(r) r^2 \sin^2 \theta dt d\varphi. \quad (2.1)$$

Throughout this paper, the polytropic equation of state is assumed:

$$p = K \rho^{1+\frac{1}{N}}, \quad (2.2)$$

where  $p$  and  $\rho$  denote the pressure, mass-energy density, respectively. Here  $N$  and  $K$  are constants.

We consider oscillation modes in rotating relativistic stars such that the eigenfunctions are stationary and are composed of only one axial parity component in the limit  $\Omega \rightarrow 0$ . Subclass of those modes should be a relativistic counterpart of  $r$ -modes, which are able to oscillate in all slowly rotating Newtonian fluid stars. According to Lockitch et al. [15], such modes are allowed to exist only if the star has non-isentropic structures. Therefore, we assume stars to be non-isentropic, although the effects due to deviation from isentropic structure on oscillation modes do not appear in the first order in  $\Omega$ . According to the formalism by Lockitch et al. [15] (see, also, Ref. [11]), let us write down the pulsation equation for relativistic  $r$ -modes with accuracy up to the first order in  $\Omega$ . The metric perturbation  $\delta g_{\alpha\beta}$  and the Eulerian changes of the fluid velocity  $\delta u^\alpha$  that do not vanish in the limit  $\Omega \rightarrow 0$  are given as

$$(\delta g_{t\theta}, \delta g_{t\varphi}) = i h_{0,l}(r) \left( -\frac{\partial_\varphi Y_{lm}(\theta, \varphi)}{\sin \theta}, \sin \theta \partial_\theta Y_{lm}(\theta, \varphi) \right) e^{i\sigma t}, \quad (2.3)$$

$$(\delta u^\theta, \delta u^\varphi) = \frac{i U_l(r)}{r^2} \left( -\frac{\partial_\varphi Y_{lm}(\theta, \varphi)}{\sin \theta}, \frac{\partial_\theta Y_{lm}(\theta, \varphi)}{\sin \theta} \right) e^{i\sigma t}, \quad (2.4)$$

where  $Y_{lm}(\theta, \varphi)$  are the usual spherical harmonic functions, and  $\sigma$  denotes oscillation frequency measured in the inertial frame at the spatial infinity. All other perturbed quantities become higher order in  $\Omega$ . Note that this form of eigenfunctions is the same as that for zero-frequency modes in a spherical non-isentropic star, because  $r$ -modes become zero-frequency ones in the limit  $\Omega \rightarrow 0$  [20]. The metric perturbation  $h_{0,l}$  obeys a second order ordinary differential equation,

$$D_{lm}(r; \bar{\sigma}) \left[ e^{\nu-\lambda} \frac{d}{dr} \left( e^{-\nu-\lambda} \frac{dh_{0,l}}{dr} \right) - \left( \frac{l(l+1)}{r^2} + \frac{-2+2e^{-2\lambda}}{r^2} + 8\pi(p+\rho) \right) h_{0,l} \right] + 16\pi(p+\rho)h_{0,l} = 0, \quad (2.5)$$

where

$$D_{lm}(r; \bar{\sigma}) \equiv 1 - \frac{2m\bar{\omega}}{l(l+1)\bar{\sigma}}, \quad (2.6)$$

Here, we have introduced the effective rotation angular velocity of fluid,

$$\bar{\omega} = \Omega - \omega, \quad (2.7)$$

and the corotating oscillation frequency,

$$\bar{\sigma} = \sigma + m\Omega. \quad (2.8)$$

The velocity perturbation of fluids  $U_l$  is determined from the function  $h_{0,l}$  through the following algebraic relation,

$$\left[ 1 - \frac{2m\bar{\omega}}{l(l+1)\bar{\sigma}} \right] U_l + h_{0,l} = 0. \quad (2.9)$$

Equations (2.5) and (2.9) are our basic equations, which were derived by Kojima [11] first. Note that the equations (2.5) and (2.9) lose their meaning in the  $l = 0$  case, because there are no axial modes with  $l = 0$ .

Because equations (2.5) are second order ordinary differential equations, two boundary conditions are required to determine solutions uniquely. From regularity of physical quantities at  $r = 0$  the function  $h_{0,l}$  must vanish at the center of a star. This condition is explicitly given as

$$r \frac{dh_{0,l}}{dr} - (l+1) h_{0,l} = 0 \quad \text{as } r \rightarrow 0. \quad (2.10)$$

Outside the star equations (2.5) have general solutions as follows:

$$h_{0,l}(r) = A \sum_{k=0}^{\infty} a_k r^{-l-k} + B \sum_{k=0}^{\infty} b_k r^{l+1+k}, \quad (2.11)$$

where  $A$  and  $B$  are arbitrary constants. Here  $a_k$  and  $b_k$  are constants determined from recurrence relations although explicit expressions are omitted here. Note that  $a_k$  and  $b_k$  do not depend on frequency  $\bar{\sigma}$ . In most previous studies the condition  $B = 0$  have been chosen as a boundary condition at the spatial infinity because spacetime must be regular everywhere. In this paper we call this condition “proper boundary condition”. This condition means the spacetime is asymptotically flat. Therefore, solutions satisfying the condition  $B = 0$  can not describe any gravitational radiation emission from a star at all. In order to include effect of gravitational radiation emission in the solutions, we must not require to demand the zero for  $B$ . This was first shown and used in the study for post-Newtonian approximation by Thorne [18].

Next, let us consider the boundary conditions for quasi-normal mode solutions, according to similar consideration to that by Thorne [18] (see, also, Ref. [21]). In the derivation of our basic equations slow motion approximation is necessarily used as well as slow rotation one because the frequency of oscillation restored by Coriolis force is the same order of the stellar rotation frequency. This slow motion approximation must nicely work near the star as long as low frequency oscillations are considered. Therefore, higher order time derivative of perturbed quantities are neglected in governing equations (2.5). If an oscillating star emits gravitational radiation, however, some of such omitted terms must become important in the radiation zone because a term such as  $\sigma r$  becomes dominant among all terms in the governing equations. Besides, rotational effects due to the stellar rotation fall off faster than  $1/r^2$  as  $r \rightarrow \infty$ . Thus, such slow motion approximation is not good far from the star, if gravitational wave are radiated from the stars. Regge-Wheeler equations with correction terms due to the stellar rotation, in fact, govern the axial perturbations sufficiently far from the star even when low frequency oscillations like  $r$ -modes are induced. Since  $\sigma r \gg 1$  in radiation zone, Regge-Wheeler equations can be approximately written as

$$r^2 \frac{d^2 X_l(r)}{dr^2} + \sigma^2 r^2 X_l(r) - l(l+1) X_l(r) + O\left(\frac{M}{r}\right) = 0, \quad (2.12)$$

where  $X_l(r)$  are Regge-Wheeler’s functions. Here  $M$  denotes the gravitational mass of the star. The metric functions  $h_{0,l}$  are determined from the functions  $X_l$  by the equation:

$$h_{0,l} = \frac{d(r X_l(r))}{dr} + O\left(\frac{M}{r}\right), \quad (2.13)$$

(see, e.g. Refs [22,23]). Here we have considered the limiting case when  $M/r \ll 1$  for simplicity. The general solutions to equations (2.12) can be given analytically:

$$X_l(\sigma r) = \sigma r (C j_l(\sigma r) + D n_l(\sigma r)), \quad (2.14)$$

where  $j_l$  and  $n_l$  are spherical Bessel functions and  $C$  and  $D$  are arbitrary constants. The asymptotic forms in the radiation zone, that is, when  $\sigma r \gg 1$ , are given as

$$X_l(\sigma r) \sim C \cos \left[ \sigma r - \frac{1}{2}(l+1)\pi \right] + D \sin \left[ \sigma r - \frac{1}{2}(l+1)\pi \right]. \quad (2.15)$$

Now we are interested in quasi-normal modes of a star. Thus, the no incoming radiation conditions are required for metric perturbations. From equation (2.15) it is found that this condition becomes the relation  $D = -iC$ . For this choice of constants the asymptotic solutions reduce to the form as:

$$X_l(\sigma r) e^{i\sigma t} \sim C \exp \left[ i\sigma(t-r) + \frac{i}{2}(l+1)\pi \right], \quad \text{as } \sigma r \rightarrow \infty. \quad (2.16)$$

Since the frequencies of the oscillations restored by the Coriolis force are proportional to the rotational frequency  $\Omega$ ,  $\sigma^2 R^2 \approx \Omega^2 R^2 = \epsilon^2 M/R$ , where  $\epsilon$  denotes a small parameter for stellar rotation defined as  $\epsilon = \Omega/(M/R^3)^{1/2}$ . Here,  $R$  is the radius of the star. Thus, the surface of the star is approximately in the near zone, that is,  $\sigma r \ll 1$ , if the stellar rotation is sufficiently slow or the stellar gravity is sufficiently weak. In this approximation, solutions (2.14) can be written as

$$X_l(\sigma r) \sim -iC \frac{(2l-1)!!}{(\sigma r)^l} \left[ 1 + i \frac{(\sigma r)^{2l+1}}{(2l-1)!!(2l+1)!!} \right], \quad (2.17)$$

where the constraint  $D = -iC$  for the no incoming radiation has been used. The approximate solutions above may be valid near the stellar surface because  $M/r < 1$  is well satisfied at the stellar surface and outside the star for typical neutron star models. In the near zone, thus, the expressions for metric perturbations  $h_{0,l}$  outside the star can be given by equations (2.13) and (2.17) as follows:

$$h_{0,l} \sim C' \frac{1}{r^l} \left[ 1 + i \frac{(l+2)(\sigma r)^{2l+1}}{(l-1)(2l+1)[(2l-1)!!]^2} \right], \quad (2.18)$$

where  $C'$  is an arbitrary constant. As the outer boundary condition we require this solution to connect smoothly to interior solution obtained by solving equation (2.5) at the stellar surface. Thus, the boundary condition is explicitly given as

$$\begin{aligned} & \left[ 1 + i \frac{(l+2)(\sigma R)^{2l+1}}{(l-1)(2l+1)[(2l-1)!!]^2} \right] r \frac{dh_{0,l}}{dr}(R-x) \\ & + \left[ l - i \frac{(l+1)(l+2)(\sigma R)^{2l+1}}{(l-1)(2l+1)[(2l-1)!!]^2} \right] h_{0,l}(R-x) = 0, \quad \text{as } x \rightarrow 0, \end{aligned} \quad (2.19)$$

where we need actual values of rotation frequency  $\Omega$  to obtain eigensolutions. In this paper we assume the value of  $\Omega$  as  $\Omega = (\pi\bar{\rho})^{1/2}$ , where  $\bar{\rho}$  is the average density defined by  $\bar{\rho} = M/(4\pi R^3/3)$ . This  $\Omega$  is an approximate value for maximum rotation frequency to be possible to settle down uniformly rotating stars in hydrostatic equilibrium. In this paper, we call the condition (2.19) ‘‘near zone boundary condition’’. This method is a crude version of matched asymptotic expansions. If we consider a non-rotating limit, that is,  $\sigma = 0$  limit, the boundary condition (2.19) becomes approximation of proper boundary conditions in which only the lowest order terms in  $M/R$  are included. As shown by Lindblom et al. [21], the boundary condition similar to that by using asymptotic solution (2.18) can give a good approximate value of eigenfrequency even for  $f$ -mode oscillation although  $f$ -mode is not a low frequency oscillation mode.

### III. NUMERICAL RESULTS

As shown in the previous studies [11,12,15], we should distinguish two cases in treating equation (2.5) when the proper boundary condition is imposed. One case is regular eigenvalue problem and the other singular one. For regular eigenvalue problem equation (2.5) may have discrete mode frequency in a range,

$$\frac{2m\bar{\omega}(R)}{l(l+1)} < \bar{\sigma} \leq \frac{2m\bar{\omega}(\infty)}{l(l+1)} = \frac{2m\Omega}{l(l+1)}. \quad (3.1)$$

On the other hand, equation (2.5) becomes singular eigenvalue problem if  $\bar{\sigma}$  is in a region,

$$\frac{2m\bar{\omega}(0)}{l(l+1)} \leq \bar{\sigma} \leq \frac{2m\bar{\omega}(R)}{l(l+1)}. \quad (3.2)$$

Notice that the range (3.2) is a continuous part of spectrum of equation (2.5). As pointed out by Lockitch et al. [15] (see also, Ref. [12]) equation (2.5) becomes that of regular eigenvalue problem when the corresponding frequency has non-zero imaginary part. Thus frequency ranges above may not have clear mathematical meanings when frequency has non-zero imaginary part. According to frequency ranges above, however, we will divide eigensolutions into three classes: The first and the second class solutions are characterized by their real part of frequency satisfying inequality (3.1) and (3.2), respectively. The third class composes of a compensative set of the first and the second class.

First of all, we concentrate our attention to  $r$ -mode solutions with frequency whose real part is in a range (3.1). We compute frequencies of mode solutions for several polytropic stellar models. In the present study, only the fundamental  $r$ -modes, whose eigenfunction  $U_m$  has no node in radial direction except at stellar center, are obtained. This is similar to that in studies for the proper boundary condition case [16,17]. In Figures 1 and 2, real and imaginary parts of scaled eigenfrequencies  $\kappa \equiv \bar{\sigma}/\Omega$  of  $r$ -modes are, respectively, given as functions of  $M/R$ . Eigenfrequencies for stars with four different polytropic indices,  $N = 0, 0.5, 0.75$ , and  $1$ , are shown in panels in both figures, respectively. Only frequency curves for the modes with  $l = m = 2$  are depicted along a relativistic factor  $M/R$  because they are considered to be the most important modes for  $r$ -mode instability.

Real parts of frequency illustrated in Figure 1 are in good agreement with Figure 1 of Ref. [16], in which frequency are obtained by imposing the proper boundary condition for asymptotically flat spacetime. The relative differences are less than 0.1%. This shows that our approximation works nicely and higher order effect of  $M/r$  on the outer boundary condition is not so important for the determination of real parts of frequency. We also find that frequency curves in Figure 1 are terminated at some value of  $M/R$  beyond which equilibrium states can still exist. Here, the maximum values of  $M/R$  for polytropic equilibrium stars having  $N = 0, 0.5, 0.75$ , and  $1.0$  are given by  $4/9, 0.385, 0.349$ , and  $0.312$ , respectively. It is also found that length of frequency curves tend to be short as a polytrope index  $N$  increases. This feature is similar to that for the case where the proper boundary condition is used. And those terminal points of frequency curves appear at almost the same values of relativistic factors as those for the proper boundary condition case (see Ref. [16]). Beyond the value of the relativistic factors corresponding to those terminal points, we can obtain a lot of eigensolutions with a singular eigenfunction but not with a regular one. Furthermore, the real part of the corresponding frequency belongs to a range (3.2) but not a range (3.1).

From Figure 2, we can see that  $r$ -modes obtained in this study are all unstable. It is also found that curves for imaginary parts of  $\kappa$  have one relative minimum near the terminal point of frequency curves. Those minimum values are given as  $\text{Im}(\kappa(M/R = 0.425)) = -2.9 \times 10^{-3}$ ,  $\text{Im}(\kappa(M/R = 0.344)) = -6.4 \times 10^{-4}$ ,  $\text{Im}(\kappa(M/R = 0.258)) = -1.3 \times 10^{-4}$ , and  $\text{Im}(\kappa(M/R = 0.095)) = -2.2 \times 10^{-6}$ , for stars having  $N = 0, 0.5, 0.75$ , and  $1.0$ , respectively. Values of  $\text{Im}(\kappa)$  also approach zero as relativistic factor  $M/R$  is getting closer to a value corresponding to that for the terminal point of frequency curves. Those behaviors of  $\text{Im}(\kappa)$  can be understood from the distribution of eigenfunctions  $U_l(r)$  because  $\text{Im}(\kappa)$  is approximately proportional to the square of the current multipole moment. In Figure 3 and 4 distributions of the eigenfunction  $U_m$  are shown for  $N = 0.5$  polytropic models having  $M/R = 0.1$  and  $M/R = 0.37$ , respectively. As seen from these figures the motion of perturbed fluid elements is strongly confined near the stellar surface when the mode frequency is getting closer to that of the terminal frequency, which satisfies  $\bar{\sigma} \approx 2m\bar{\omega}(R)/(l(l+1))$ . We can easily understand this behavior from equation (2.9). Consequently the values of current multipole moments of such modes may become small when the value of relativistic factor increases. On the other hand, the efficiency of the gravitational radiation emission becomes good as increase of the value of the relativistic factor. Due to both effects above a relative minimum of imaginary part of frequency may appear. We should notice that in  $N = 0$  case, a value of  $\text{Im}(\kappa)$  do not approach zero even when  $M/R \sim 4/9$ , as we can see from Figure 2. The reason is that the eigenfunctions  $U_m$  does not have so strong peak at the stellar surface even when  $M/R \sim 4/9$  because a relation  $\bar{\sigma} \approx 2m\bar{\omega}(R)/(l(l+1))$  is not satisfied in this case.

Next let us estimate the instability timescale for the gravitational radiation driven instability of  $r$ -mode solutions with frequency in a range (3.1). Here a typical neutron star model whose mass and radius are respectively  $1.4M_\odot$  and  $12.57$  km is considered for  $N = 0, 0.5$ , and  $0.75$  polytropic models. When the star rotates with the angular frequency  $\Omega = (3/4 \times GM/R^3)^{1/2} = 8377 \text{ s}^{-1}$ , growing timescales  $\tau_j$  of the  $r$ -mode instability are given as  $\tau_j = 1.29 \text{ s}, 2.04 \text{ s}$ , and  $2.97 \text{ s}$  for  $N = 0, 0.5$ , and  $0.75$  models, respectively. Note that for  $N = 1$  case the growing timescale cannot be estimated because we can not find discrete  $r$ -mode solution for that model. These timescale are similar to those obtained from Newtonian estimate. (see, e.g. Ref. [6])

When we concentrate our attention to solutions whose frequency has real part in a frequency range (3.2), we obtain a large number of solutions whose eigenfunction has singular behavior in its real part at a radius determined by a solution of  $D_{lm}(r; \text{Re}(\bar{\sigma})) = 0$ . Besides, those solutions have severe truncation error due to a finite difference method.

Similar behavior of solutions appears in oscillations of differentially rotating disks. (see, for example, Ref. [24]) As discussed by Schutz and Verdaguer [24], this is considered a sign of the existence of continuous parts of spectrum, although the existence has to be proved by other mathematical techniques because exact continuous spectrum is never obtained from a simple numerical analysis. In the present case, we are sure that the appearance of a continuous spectrum is plausible because the imaginary parts of frequency are too small to change drastically the character of the solutions derived from the proper boundary condition. Thus, our numerical results suggest the existence of a continuous part of spectrum in Kojima's equation even when their frequency becomes complex number. As for regular solutions with frequency in a range (3.2), we cannot obtain such a solution at all. Finally we consider the third class of solutions, whose real part of frequency is neither in a range (3.1) nor (3.2). In this region of real part of frequency, we cannot obtain any solutions at all.

#### IV. DISCUSSION AND CONCLUSION

In this paper, we have investigated the properties of  $r$ -mode instability in slowly rotating relativistic polytropes. Inside the star slow rotation and the low frequency formalism that was mainly developed by Kojima [11] and Lockitch et al. [15] is employed to study axial oscillations restored by Coriolis force. At the stellar surface, in order to take account of gravitational radiation reaction effect, we use a near-zone boundary condition, which was devised by Thorne [18] and recently developed for relativistic pulsations by Lindblom et al. [21], instead of the usually imposed boundary condition for asymptotically flat spacetime. Due to the boundary condition, complex frequencies whose imaginary part represents secular instability are obtained for  $r$ -mode oscillations in some polytropic models. It is found that such discrete  $r$ -mode solutions can be obtained only for some restricted polytropic models. Basic properties of mode solutions of equation (2.5) that obtained in this study is similar to those with the boundary condition for asymptotically flat spacetime although its frequency becomes complex because of the near-zone boundary condition.

As suggested by Lockitch et al. [15] (see, also Refs [12,16,17]), when an eigenfrequency becomes complex number, which expresses the damping of the oscillation due to the energy dissipation such as the gravitational radiation emission, there is a possibility that the existence of the continuous part of spectrum in the eigenfrequency of equation (2.5) is avoided. In this study, we consider the complex frequency corresponding to the quasi-normal mode as mode solutions of equation (2.5) by imposing the lowest order near-zone boundary condition. Our numerical results suggest the existence of a continuous part of spectrum in Kojima's equation even when the frequency is allowed to be a complex number. However, we still think that the existence of a continuous part of spectrum in axial oscillations restored by Coriolis force is not plausible because such property does not appear in Newtonian  $r$ -modes. The existence of a continuous part of spectrum in this study might be artifact due to the approximation because our treatment is the lowest order approximation. In other words, inclusion of full effect of gravitational radiation emission might avoid the existence of a continuous part of spectrum. Another possibility to prevent the appearance of continuous spectrum might be to avoid a singular reduction in the order of the equation. As Kojima and Hosonuma [14] showed, basic equations for  $r$ -mode oscillations become a fourth order ordinary differential equation for metric perturbation  $h_{0,l}$  when rotational effect up to the third order of  $\Omega/(M/R^3)^{1/2}$  is consistently considered. In this equation, extra two degrees of freedom of solutions may be used to avoid singular behavior of eigenfunction. Due to these extra boundary conditions, all eigenfrequency may become discrete. Verification of those possibilities remains future studies.

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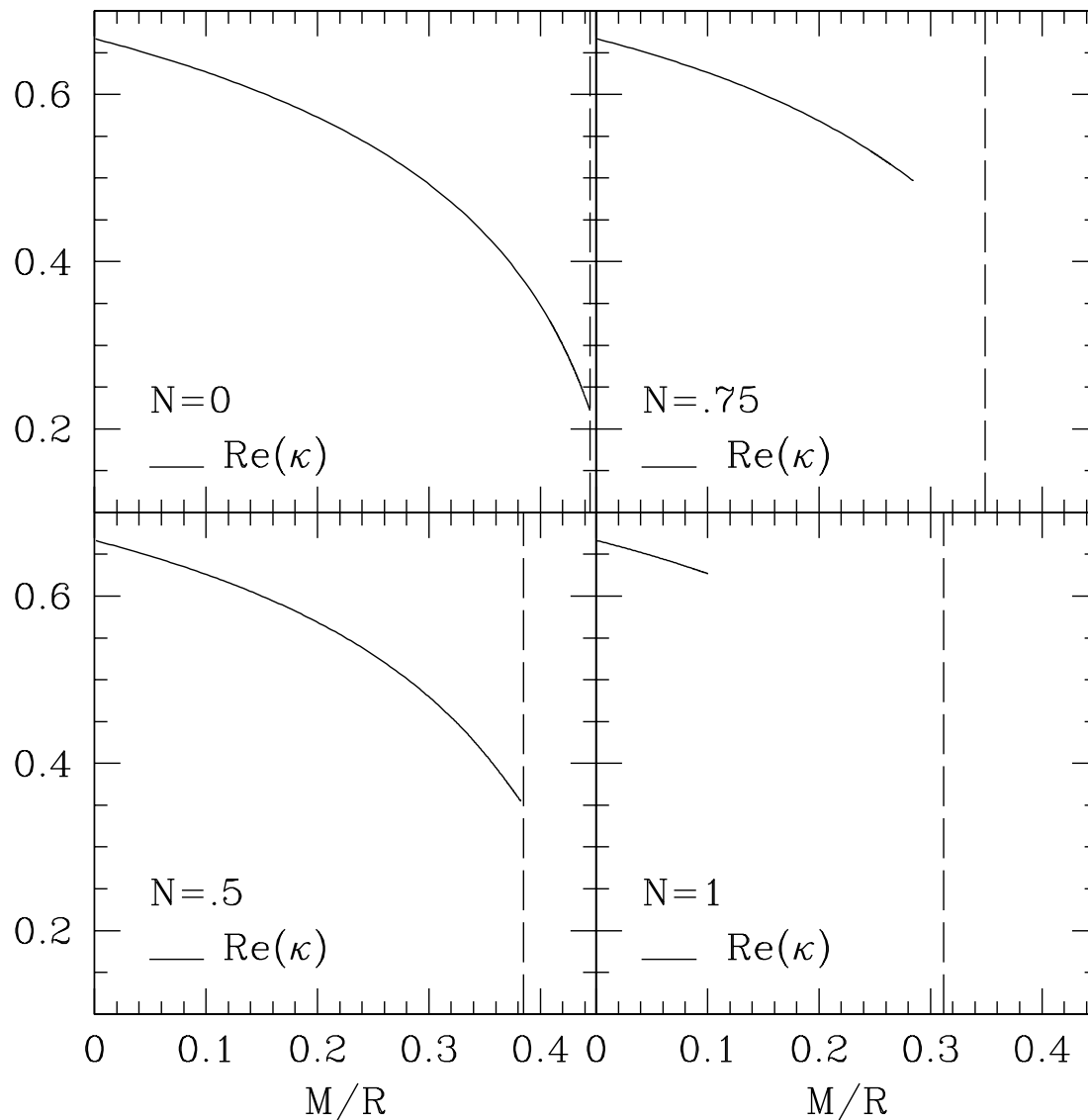


FIG. 1. Real parts of scaled frequencies  $\kappa = \bar{\sigma}/\Omega$  of the  $r$ -modes with  $l = m = 2$  plotted as functions of  $M/R$ . In each panel, the frequencies of modes for polytropic models with  $N = 0, 0.5, 0.75$ , and  $1$  are respectively shown. The labels indicating their polytropic indices  $N$  are attached in corresponding panels. Vertical dotted lines show the maximum values of  $M/R$  for equilibrium states:  $M/R = 0.444$  for  $N = 0$ ,  $M/R = 0.385$  for  $N = 0.5$ ,  $M/R = 0.349$  for  $N = 0.75$ , and  $M/R = 0.312$  for  $N = 1.0$ .



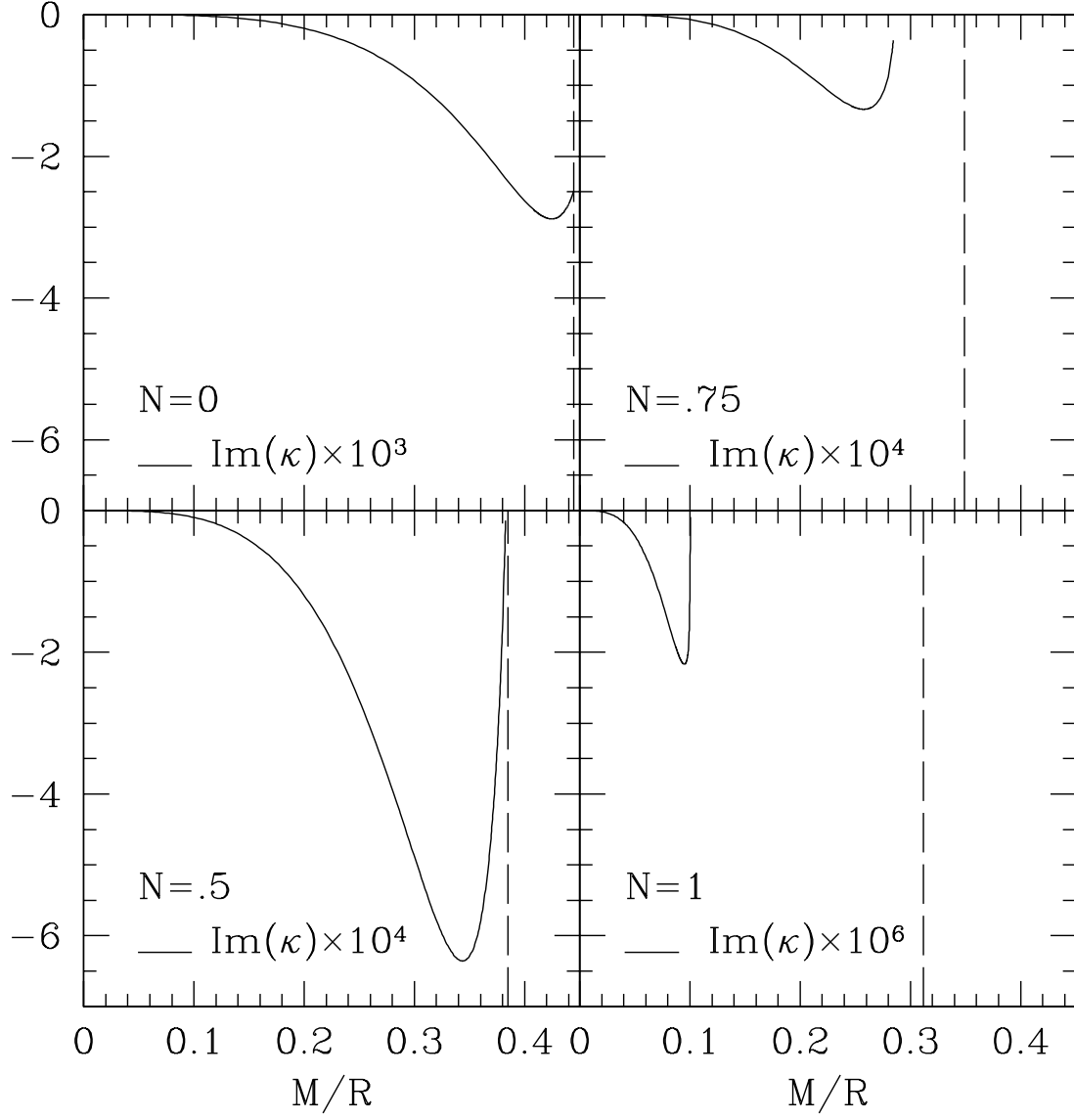


FIG. 2. The same as Figure 1 but for imaginary parts of frequency.

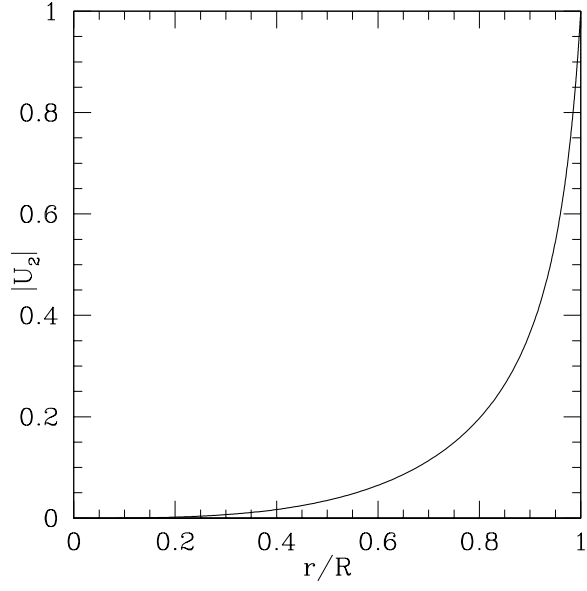


FIG. 3. Absolute value of eigenfunction  $U_2$  of a  $r$ -mode with  $l = m = 2$  for a  $N = 0.5$  polytrope with  $M/R = 0.1$  is given as a function of  $r/R$ , where normalization of the eigenfunction is given as  $U_2(R) = 1$ .

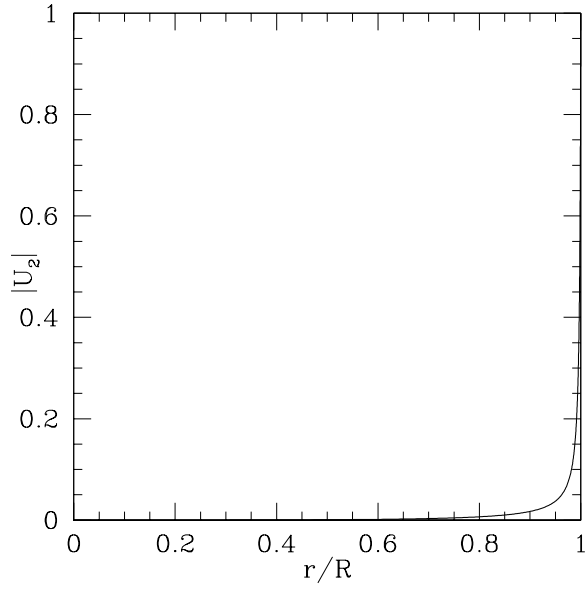


FIG. 4. The same as Figure 3 but for  $M/R = 0.37$ .